

ONE-DIMENSIONAL F -DEFINABLE SETS IN $F((t))$

WILL ANSCOMBE

ABSTRACT. We study definable sets in power series fields with perfect residue fields. We show that certain ‘one-dimensional’ definable sets are in fact existentially definable. This allows us to apply results from [2] about existentially definable sets to one-dimensional definable sets.

More precisely, let F be a perfect field and let \mathbf{a} be a tuple from $F((t))$ of transcendence degree 1 over F . Using the description of F -automorphisms of $F((t))$ given by Schilling, in [9], we show that the orbit of \mathbf{a} under F -automorphisms is existentially definable in the ring language with parameters from $F(t)$.

We deduce the following corollary. Let X be an F -definable subset of $F((t))$ which is not contained in F , then the subfield generated by X is equal to $F((t^{p^n}))$, for some $n < \omega$.

Let F be a fixed **perfect** field and let v denote the t -adic valuation on the power series field $F((t))$. The valuation ring of v is $F[[t]]$ and the maximal ideal is $tF[[t]]$. Let \mathcal{U} denote the set of uniformisers in $F((t))$, i.e. those elements of value 1.

Let $\mathcal{L}_{\text{ring}} := \{+, \cdot, 0, 1\}$ be the *language of rings* and let $\mathcal{L}_{vf} := \mathcal{L}_{\text{ring}} \cup \{O\}$ be the *language of valued fields*, which is an expansion of $\mathcal{L}_{\text{ring}}$ by a unary predicate O (intended to be interpreted as the valuation ring). Let $\mathcal{L}_{\text{ring}}(F)$ and $\mathcal{L}_{vf}(F)$ denote the expansion of each language by constants for elements of F . For a tuple $\mathbf{a} \subseteq F((t))$, we let $\text{Orb}(\mathbf{a})$ denote the orbit of \mathbf{a} under the $\mathcal{L}_{vf}(F)$ -automorphisms of $F((t))$, and let $\text{tp}(\mathbf{a})$ denote the $\mathcal{L}_{vf}(F)$ -type of \mathbf{a} . In a slight abuse of notation, we write $(F((t)), v)$ in place of the \mathcal{L}_{vf} -structure $(F((t)), F[[t]])$. Let p be the characteristic exponent of F , i.e if $\text{char}(F) > 0$ then $p := \text{char}(F)$, and otherwise $p := 1$.

The well-known theorem of Ax-Kochen/Ershov (see for example Theorem 3, [4]) gives an axiomatisation of the \mathcal{L}_{vf} -theory of $(F((t)), v)$ in the case that $\text{char}(F) = 0$. However, there is no corresponding known axiomatisation for the theory of $(F((t)), v)$ if $\text{char}(F) > 0$; neither is there a description of the definable sets in this structure. This note provides a small step forward by studying the ‘one-dimensional’ F -definable subsets of $F((t))$, i.e. those F -definable sets which contain a tuple of transcendence degree 1 over F . Since the $\text{char}(F) = 0$ case is so well-understood, the reader might like to focus on the case $\text{char}(F) > 0$, although our results hold for arbitrary characteristic. In section 5 we prove the following theorem.

Comments very welcome! March 20, 2015.

This research forms part of the author’s doctoral thesis, [1], completed under the supervision of Jochen Koenigsmann and supported by EPSRC. This note was prepared while the author was funded by EPSRC grant EP/K020692/1.

Theorem 1. *Let \mathbf{a} be a tuple from $F((t))$ of transcendence degree 1 over F . Then $\text{Orb}(\mathbf{a})$ is*

- (1) $\exists\text{-}\mathcal{L}_{\text{ring}}(F(t))$ -definable (i.e. definable by an existential $\mathcal{L}_{\text{ring}}(F(t))$ -formula),
- (2) $\mathcal{L}_{\text{ring}}(F)$ -definable, and
- (3) equal to the type $\text{tp}(\mathbf{a})$ of \mathbf{a} over F .

By combining this with work from [2], in section 6 we are able to deduce the following corollary.

Corollary 2. *Let $X \subseteq F((t))$ be an $\mathcal{L}_{vf}(F)$ -definable subset. Then either $X \subseteq F$ or there exists $n < \omega$ such that*

$$(X) = F((t^{p^n})),$$

where (X) denotes the subfield of $F((t))$ generated by X .

Finally, we give corollaries about subfields generated by \mathcal{L}_{vf} -definable subsets of $\mathbb{F}_p((t))$ and $\mathbb{F}_p((t))^{\text{perf}}$.

1. F -AUTOMORPHISMS OF $F((t))$

Schilling gives, in [9], a description of the $\mathcal{L}_{vf}(F)$ -automorphisms of $F((t))$, and their representation as substitutions $t \mapsto s$ for $s \in \mathcal{U}$. In Lemma 1 of [9], Schilling shows that all $\mathcal{L}_{\text{ring}}$ -automorphisms are in fact \mathcal{L}_{vf} -automorphisms. Let \mathbf{G} denote the group of $\mathcal{L}_{vf}(F)$ -automorphisms of $F((t))$. For $b \in F((t))$, let $\text{Orb}(b)$ denote the orbit of b under the action of \mathbf{G} .

Fact 3. [Theorem 1, [9]] *Let $\circ : F((t)) \times \mathcal{M} \rightarrow F((t))$ denote the composition map. It is continuous map. The restriction of \circ to $\mathcal{U} \times \mathcal{U}$ is associative, t is the identity element, and every element is invertible. For each $s \in \mathcal{M}$, the map given by $x \mapsto x \circ s$ is a ring homomorphism. Thus (\mathcal{U}, \circ) is a group which acts on $F((t))$ as a group of F -automorphisms. The corresponding representation $(\mathcal{U}, \circ) \rightarrow \mathbf{G}$ is an isomorphism.*

In particular, we have the following.

Fact 4. $\mathcal{U} = \text{Orb}(t)$.

For $n > 1$, let \mathbf{G}_n denote the subgroup of \mathbf{G} of those automorphisms corresponding to substitutions $t \mapsto s$, for $s \in t + \mathcal{M}^n$. In Theorem 3 of [9], Schilling proves that these groups are the same as the *pseudo-ramification groups* of MacLane, see Section 9 of [6]. For $b \in F((t))$ and $n > 1$, let $\text{Orb}_n(b)$ denote the orbit of b under the action of \mathbf{G}_n .

Recall that $f \in F[[t]]$ may also be thought of as a function

$$\begin{aligned} f : tF[[t]] &\longrightarrow F[[t]] \\ x &\longmapsto f(x) := f \circ x \end{aligned}$$

Fact 5. *Let $f \in F[[t]]$ and let $n > 1$. Then $f(t + \mathcal{M}^n) = \text{Orb}_n(f(t))$.*

2. A HENSEL-LIKE LEMMA

In this section we prove a ‘Hensel-like’ Lemma (Proposition 6) in the ring $F[[t]]$ of formal power series over an *arbitrary* field F . Proposition 6 can be deduced from a version of Newton’s Lemma for power series, but we give a direct proof. For $N \in \mathbb{Z}$, let

$$B(N, a) := \{b \in F((t)) \mid v(b - a) > N\}$$

denote the *open ball of radius m around a* . For tuples $\mathbf{N} = (N_1, \dots, N_s) \subseteq \mathbb{Z}$ and $\mathbf{a} = (a_1, \dots, a_s) \subseteq F((t))$, we write

$$\mathbf{B}(\mathbf{N}; \mathbf{a}) := B(N_1; a_1) \times \dots \times B(N_s; a_s).$$

Proposition 6. *Suppose that $f \in F[[t]] \setminus F[[t]]^p$. Then, for each $n < \omega$, there exists $N < \omega$ such that*

$$B(N; f(t)) \subseteq f(t + \mathcal{M}^n).$$

The rough idea is as follows. Let $(y_j)_{j \geq 1}$ be ‘formal indeterminates’ over F (e.g. algebraically independent over F in a field extension linearly disjoint from $F((t))/F$) and let $y := \sum_{j \geq 1} y_j t^j$. We study the power series

$$f(y) \in F[(y_j)_{j \geq 1}][[t]].$$

The coefficients of $f(y)$ are polynomials in finitely many of the variables (y_j) . We show that the h -th coefficient is a polynomial in the variables $(y_1, \dots, y_{h'})$, where the function $h \mapsto h'$ is eventually strictly increasing. This allows us to choose N so that we may recursively define solutions to the equations $f(y) = b$, for $b \in B(N; f(t))$.

Without further comment, we shall assume that f, b, y are written as:

$$f = \sum_i a_i t^i, \quad b = \sum_h b_h t^h, \quad y = \sum_{j \geq 1} y_j t^j.$$

For $i < \omega$, we write $y^i = \sum_j y_j^{(i)} t^j$.

Lemma 7. *Let $i, j < \omega$ be such that $p \nmid i$. Then there exists $Y_j^{(i)} \in \mathbb{F}[y_1, \dots, y_{j-i}]$ such that*

$$y_j^{(i)} = Y_j^{(i)} + i y_1^{i-1} y_{j-i+1}.$$

In particular, $y_j^{(i)} \in \mathbb{F}[y_1, \dots, y_{j-i+1}]$.

Proof. We have that

$$y_j^{(i)} = \sum_{\sum_{r=1}^i j_r = j} \left(\prod_{r=1}^i y_{j_r} \right).$$

We observe that $y_j^{(i)}$ is a polynomial in the variables $(y_j)_{j < \omega}$. The variable with the highest index that occurs nontrivially is y_{j-i+1} and the only term in which y_{j-i+1} appears is $i y_1^{i-1} y_{j-i+1}$. \square

Lemma 8. *Let $i, j, k, l < \omega$ be such that $i = kp^l$ and $p \nmid k$. Then*

$$y_j^{(i)} = \begin{cases} 0 & \text{if } p^l \nmid j \\ Y_{jp^{-l}}^{(k)} + ky_1^{k-1}y_{jp^{-l}-k+1} & \text{if } p^l \mid j. \end{cases}$$

In particular, $y_j^{(i)} \in \mathbb{F}[y_1, \dots, y_{jp^{-l}-k+1}]$.

Proof. First we note that $y_j^{(i)} = y_{jp^{-l}}^{(k)}$. Then the conclusion is immediate from Lemma 7. \square

Lemma 8 motivates the study of the functions $h \mapsto hp^{-l} - k + 1$.

Definition 9. Let $i_0 := \min\{i \mid a_i \neq 0 \text{ and } p \nmid i\}$ and let

$$N' := \left\lceil \max \left\{ \frac{i_0 - k}{1 - p^{-1}} \mid k < i_0 \right\} \right\rceil.$$

Note that i_0 is **not** a valuation and it is well-defined by our assumption that $f \notin F[[x]]$.

Lemma 10. *Let $h < \omega$ be such that $N' < h$; and let $i < \omega$ be such that $i \neq i_0$ and $a_i \neq 0$. Choose $k, l < \omega$ such that $i = kp^l$ and $p \nmid k$. Then we have*

$$hp^{-l} - k + 1 < h - i_0 + 1.$$

Proof. First, suppose that $k < i_0$. Then we must have $0 < l$, by definition of i_0 . We have

$$\frac{i_0 - k}{1 - p^{-1}} \leq N' < h.$$

A simple rearrangement gives:

$$hp^{-1} - k + 1 < h - i_0 + 1.$$

Thus

$$\begin{aligned} hp^{-l} - k + 1 &\leq hp^{-1} - k + 1 \\ &< h - i_0 + 1, \end{aligned}$$

as required. On the other hand, suppose that $i_0 \leq k$. Then

$$\begin{aligned} hp^{-l} - k + 1 &\leq h - k + 1 \\ &\leq h - i_0 + 1. \end{aligned}$$

It is clear that equality holds if and only if $i_0 = k$ and $l = 0$; i.e. if and only if $i = i_0$. \square

For $h < \omega$, let $C_h(\sum_i a_i t^i) := a_h$.

Lemma 11. *Let $h < \omega$ be such that $h > N'$. Then there exists $Z_h \in \mathbb{F}(a_i)_{i < \omega}[y_1, \dots, y_{h-i_0}]$ such that*

$$C_h(f(y)) = Z_h + a_{i_0}i_0y_1^{i_0-1}y_{h-i_0+1}.$$

In particular, $C_h(f(y)) \in \mathbb{F}(a_i)_{i < \omega}[y_1, \dots, y_{h-i_0+1}]$.

Proof. Combining Lemma 10 and Lemma 8, we have that $\sum_{i \neq i_0} a_i y_h^{(i)} \in \mathbb{F}(a_i)_{i < \omega}[y_1, \dots, y_{h-i_0}]$. Another application of Lemma 8 gives that there exists $Y_h^{(i_0)} \in \mathbb{F}(a_i)_{i < \omega}[y_1, \dots, y_{h-i_0}]$ such that

$$y_h^{(i_0)} = Y_h^{(i_0)} + i_0 y_1^{i_0-1} y_{h-i_0+1}.$$

Now:

$$\begin{aligned} C_h(f(y)) &= C_h(\sum_i a_i y^i) \\ &= C_h(\sum_i a_i \sum_j y_j^{(i)} t^j) \\ &= \sum_i a_i C_h(\sum_j y_j^{(i)} t^j) \\ &= \sum_i a_i y_h^{(i)} \\ &= \sum_{i \neq i_0} a_i y_h^{(i)} + a_{i_0} (Y_h^{(i_0)} + i_0 y_1^{i_0-1} y_{h-i_0+1}) \\ &= Z_h + a_{i_0} i_0 y_1^{i_0-1} y_{h-i_0+1}, \end{aligned}$$

where $Z_h := \sum_{i \neq i_0} a_i y_h^{(i)} + a_{i_0} Y_h^{(i_0)} \in \mathbb{F}(a_i)_{i < \omega}[y_1, \dots, y_{h-i_0}]$. \square

2.1. The proof of Proposition 6. Choose $N < \omega$ such that $N \geq N'$ and $N - i_0 + 1 \geq n - 1$. Let $b \in B(N; f(t))$. We seek $y \in t + \mathcal{M}^n$ such that

$$f(y) = b.$$

We rephrase this goal: we seek $(y_j)_{j < \omega} \subseteq F$ such that:

- (1) $\sum_{j < \omega} y_j t^j \in t + \mathcal{M}^n$ and
- (2) for each $h < \omega$, we have $C_h(f(\sum y_j t^j)) = b_h$.

Set

- (1) $y_1 := 1$ and
- (2) $y_i := 0$, for $i \in \{2, \dots, N - i_0 + 1\}$.

Then $\sum_{j=1}^{N-i_0+1} y_j t^j = t$. Trivially we have:

- (1) $t \in t + \mathcal{M}^n$ and
- (2) $C_h(f(t)) = b_h$, for all $h \leq N$.

We now recursively define y_j , for $j > N - i_0 + 1$. Let $H > N$ and suppose that we have defined y_j for $j < H - i_0 + 1$ such that

$$C_h(f(d)) = b_h,$$

for all $h < H$, where $d := \sum_{j=1}^{H-i_0} y_j t^j$.

By rearranging the formula in Lemma 11, we may choose $y_{H-i_0+1} \in F$ such that

$$C_H(f(e)) = b_H,$$

where $e := d + y_{H-i_0+1} t^{H-i_0+1}$. It is also clear that

$$C_h(f(e)) = C_h(f(d)) = b_h,$$

for $h < H$. Then $y := \sum_{j < \omega} y_j t^j$ is as required. This completes the proof of Proposition 6.

3. ORBITS ARE ‘NEARLY OPEN’

Lemma 12. *Let $a \in F((t)) \setminus F((t))^p$ and let $n < \omega$. Then there exists $N < \omega$ such that*

$$B(N; a) \subseteq \text{Orb}_n(a).$$

Proof. First we suppose that $a \in F[[t]]$. By applying Proposition 6 to $f := a$, there exists $N < \omega$ such that $B(N; f(t)) \subseteq f(t + \mathcal{M}^n)$. By Fact 5, $f(t + \mathcal{M}^n) = \text{Orb}_n(a)$.

If, on the other hand, $a \notin F[[t]]$ then $a^{-1} \in F[[t]]$, so there exists $N < \omega$ such that $B(N; a^{-1}) \subseteq \text{Orb}_n(a^{-1})$. Since the map $x \mapsto x^{-1}$ is continuous, there exists $N' < \omega$ such that $B(N'; a) \subseteq \text{Orb}_n(a^{-1})^{-1} = \text{Orb}_n(a)$, as required. \square

We now extend Lemma 12 to elements of $F((t))^p \setminus F$.

Lemma 13. *Let $b \in F((t)) \setminus F$ and let $n \in \mathbb{N}$. There exists $l, N < \omega$ such that*

$$B(N; b) \cap F((t))^{p^l} \subseteq \text{Orb}_n(b).$$

Proof. Let $l \in \mathbb{N}$ be such that $b \in F((t))^{p^l} \setminus F((t))^{p^{l+1}}$. Set $a := b^{p^{-l}}$. By Lemma 12, there exists $N' < \omega$ such that $B(N'; a) \subseteq \text{Orb}_n(a)$. Let $N := p^l(N' + 1) - 1$. For $x \in F((t))$ we have

$$\begin{aligned} v(x - a) > N' & \quad \text{iff} \quad v(x - a) \geq N' + 1 \\ & \quad \text{iff} \quad v((x - a)^{p^l}) \geq p^l(N' + 1) \\ & \quad \text{iff} \quad v(x^{p^l} - b) > p^l(N' + 1) - 1 = N. \end{aligned}$$

Thus $x \in B(N'; a)$ if and only if $x^{p^l} \in B(N; b)$. Therefore $B(N; b) \cap F((t))^{p^l} \subseteq \text{Orb}_n(b)$. \square

Lemma 14. *Let $c \in F((t)) \setminus F$ and let $N < \omega$. Then there exists $n < \omega$ such that $\text{Orb}_n(c) \subseteq B(N; c)$.*

Proof. This follows from the continuity of the map $u \mapsto c \circ u$. \square

4. A DESCRIPTION OF ORBITS OF ONE-DIMENSIONAL TUPLES

For an \mathbf{x} -tuple $\mathbf{a} \subseteq F((t))$, we let $\text{locus}(\mathbf{a})$ denote the $F((t))$ -rational points of the smallest Zariski-closed set which is defined over F and contains \mathbf{a} . Equivalently, $\text{locus}(\mathbf{a})$ is the set of those \mathbf{x} -tuples $\mathbf{a}' \subseteq F((t))$ which are zeroes of all polynomials (with coefficients from F) which are zero at \mathbf{a} . For $l \in \mathbb{N}$, let $P_l := \{(y, \mathbf{z}) \mid y \in F((t))^{p^l}\}$.

Lemma 15. *Let \mathbf{a} be a tuple from $F((t))$ of transcendence degree 1 over F . Then there exist $l < \omega$ and a tuple $\mathbf{N} \subseteq \omega$ such that*

$$\text{locus}(\mathbf{a}) \cap B(\mathbf{N}; \mathbf{a}) \cap P_l \subseteq \text{Orb}(\mathbf{a}).$$

Proof. Since $F((t))/F$ is separable, we may re-write the tuple \mathbf{a} as a (y, \mathbf{z}) -tuple (b, \mathbf{c}) such that \mathbf{c} is separably algebraic over $F(b)$ and b is transcendental over F ; i.e. b is a separating transcendence base for \mathbf{a} over F .

By Theorem 7.4 of [7], a field admitting a nontrivial henselian valuation (such as $F((t))$) satisfies the ‘Implicit Function Theorem’ (for polynomials). By an easy elaboration of the

Implicit Function Theorem (as given in [2]), and since \mathbf{c} is separably algebraic over $F(b)$, there exist $N_1, \mathbf{N}_2 \in \mathbb{Z}$ such that

$$\text{locus}(b, \mathbf{c}) \cap B(N_1, \mathbf{N}_2; b, \mathbf{c})$$

is the graph of a continuous function

$$B(N_1; b) \longrightarrow B(\mathbf{N}_2; \mathbf{c}).$$

By Lemma 14, we may choose $n < \omega$ so that

$$\text{Orb}_n(\mathbf{c}) \subseteq B(\mathbf{N}_2; \mathbf{c});$$

and, by Lemma 13, there exists $l, N'_1 < \omega$ such that $N'_1 \geq N_1$ and

$$B(N'_1; b) \cap F((t))^{p^l} \subseteq \text{Orb}_n(b).$$

Our aim is to show that

$$\text{locus}(b, \mathbf{c}) \cap B(N'_1, \mathbf{N}_2; b, \mathbf{c}) \cap P_l \subseteq \text{Orb}(b, \mathbf{c}).$$

The result then follows from setting $\mathbf{N} := (N'_1, \mathbf{N}_2)$.

Let $(y, \mathbf{z}) \in \text{locus}(b, \mathbf{c}) \cap B(N'_1, \mathbf{N}_2; b, \mathbf{c}) \cap P_l$. Then $y \in B(N'_1; b) \cap F((t))^{p^l} \subseteq \text{Orb}_n(b)$. Thus there exists $s \in t + \mathcal{M}^n$ (corresponding to the automorphism σ) such that $y = \sigma(b)$.

By our choice of n , we have that $\sigma(\mathbf{c}) \in B(\mathbf{N}_2; \mathbf{c})$. Thus $(y, \sigma(\mathbf{c})) \in B(N'_1, \mathbf{N}_2; b, \mathbf{c})$. Since σ is an automorphism, we also have that $(y, \sigma(\mathbf{c})) = \sigma(b, \mathbf{c}) \in \text{locus}(b, \mathbf{c})$.

Therefore both tuples $(y, \sigma(\mathbf{c}))$ and (y, \mathbf{z}) are members of $\text{locus}(b, \mathbf{c}) \cap B(N'_1, \mathbf{N}_2; b, \mathbf{c})$, which is the graph of a function. Thus $\sigma(\mathbf{c}) = \mathbf{z}$.

We have shown that

$$\begin{aligned} (y, \mathbf{z}) &= \sigma(b, \mathbf{c}) \\ &\in \text{Orb}_m(b, \mathbf{c}) \\ &\subseteq \text{Orb}(b, \mathbf{c}), \end{aligned}$$

as required. □

Lemma 16. *Let \mathbf{a} be a tuple from $F((t))$ of transcendence degree 1 over F , and choose $l < \omega$ and $\mathbf{N} \subseteq \omega$ as in Lemma 15. Then*

$$\text{locus}(\mathbf{a}) \cap \mathbf{U} \cap P_l = \text{Orb}(\mathbf{a}),$$

where \mathbf{U} is the open set $\bigcup_{\sigma \in \mathbf{G}} B(\mathbf{N}; \sigma(\mathbf{a}))$.

Proof. First we note that $\text{locus}(\mathbf{a})$ and P_l are closed set-wise under automorphisms from \mathbf{G} .

(\subseteq) This follows immediately from Lemma 15 noting that $\text{Orb}(\mathbf{a})$ is closed under automorphisms.

(\supseteq) We note that $\mathbf{a} \in \text{locus}(\mathbf{a}) \cap B(\mathbf{N}; \mathbf{a}) \cap P_l$. The result then follows by applying automorphisms. □

5. DEFINABILITY OF ORBITS OF ONE-DIMENSIONAL TUPLES

Lemma 17. *Let $B(\mathbf{N}; \mathbf{a})$ be as in Lemma 15. There exists a tuple $\mathbf{f} \subseteq F(t)$ of rational functions such that $B(\mathbf{N}; \mathbf{a}) = B(\mathbf{N}; \mathbf{f}(t))$.*

Proof. This follows from the fact that $F(t)$ is t -adically dense in $F((t))$. \square

Theorem 1. *Let \mathbf{a} be a tuple from $F((t))$ of transcendence degree 1 over F . Then $\text{Orb}(\mathbf{a})$ is*

- (1) $\exists\text{-}\mathcal{L}_{\text{ring}}(F(t))$ -definable,
- (2) $\mathcal{L}_{\text{ring}}(F)$ -definable, and
- (3) equal to the type $\text{tp}(\mathbf{a})$ of \mathbf{a} over F .

Proof. Let notation be as in Lemma 16. In particular, there is a variable y in the tuple \mathbf{x} and $P_l = \{\mathbf{x} \mid y \in F((t))^{p^l}\}$.

- (1) Let I be the ideal in $F[\mathbf{x}]$ of polynomials which are zero on \mathbf{a} . Since $F[\mathbf{x}]$ is Noetherian, there is a tuple $\mathbf{g} = (g_1, \dots, g_r)$ of polynomials which generates I . Let $\phi(\mathbf{x})$ be the formula

$$\bigwedge_{i=1}^r g_i(\mathbf{x}) = 0;$$

then $\phi(\mathbf{x})$ defines $\text{locus}(\mathbf{a})$. Note that $\phi(\mathbf{x})$ is a quantifier-free $\mathcal{L}_{\text{ring}}(F)$ -formula.

Let $\psi(\mathbf{x})$ be the formula

$$\exists w \, w^{p^l} = y;$$

then $\psi(\mathbf{x})$ defines P_l . Note that $\psi(\mathbf{x})$ is an $\exists\text{-}\mathcal{L}_{\text{ring}}$ -formula.

Our next task is to define $B(\mathbf{N}; \sigma(\mathbf{a}))$, uniformly for $\sigma \in \mathbf{G}$. Write $\mathbf{N} = (N_1, \dots, N_s)$, $\mathbf{a} = (a_1, \dots, a_s)$, $\mathbf{f} = (f_1, \dots, f_s)$, and $\mathbf{x} = (x_1, \dots, x_s)$. Then

$$\begin{aligned} B(\mathbf{N}; \mathbf{a}) &= B(N_1; a_1) \times \dots \times B(N_s; a_s) \\ &= B(N_1; f_1(t)) \times \dots \times B(N_s; f_s(t)). \end{aligned}$$

For $j \in \{1, \dots, s\}$, let $\chi_j(x; t)$ be the formula

$$\exists y_1 \dots \exists y_{(N_j+1)} \left(x - f_j(t) = \prod_{k=1}^{N_j+1} y_k \wedge \bigwedge_{k=1}^{N_j+1} C(y_k; t) \right);$$

then $\chi_j(x; t)$ defines $B(N_j; a_j)$. Let $\chi(\mathbf{x}; t)$ be the formula

$$\bigwedge_{j=1}^s \chi_j(x_j; t);$$

then $\chi(\mathbf{x}; t)$ defines $B(\mathbf{N}; \mathbf{a})$. For $\sigma \in \mathbf{G}$, we have that $\chi(\mathbf{x}; \sigma(t))$ defines $B(\mathbf{N}; \sigma(\mathbf{a}))$.

Let $G(x; t)$ be the $\exists\text{-}\mathcal{L}_{\text{ring}}(t)$ -formula which defines \mathcal{U} , as in Lemma 23. Let $\alpha(\mathbf{x}; t)$ be the formula

$$\exists u \, (G(u; t) \wedge \chi(\mathbf{x}; u));$$

then $\alpha(\mathbf{x}; t)$ defines \mathbf{U} . Note that $\alpha(\mathbf{x}; t)$ is an $\exists\text{-}\mathcal{L}_{\text{ring}}(F(t))$ -formula.

Finally, let $\beta(\mathbf{x}; t)$ be the formula

$$(\phi(\mathbf{x}) \wedge \psi(\mathbf{x}) \wedge \alpha(\mathbf{x}; t));$$

then $\beta(\mathbf{x}; t)$ defines $\text{Orb}(\mathbf{a})$, by Lemma 16. Note that $\beta(\mathbf{x}; t)$ is an $\exists\text{-}\mathcal{L}_{\text{ring}}(F(t))$ -formula.

- (2) Let $H''(u)$ be the $\mathcal{L}_{\text{ring}}$ -formula which defines \mathcal{U} , as in Lemma 26. Let $\gamma(\mathbf{x})$ be the formula

$$\exists u (H''(u) \wedge \beta(\mathbf{x}; u));$$

then $\gamma(\mathbf{x})$ defines $\text{Orb}(\mathbf{a})$. Note that $\gamma(\mathbf{x})$ is an $\mathcal{L}_{\text{ring}}$ -formula.

- (3) It is a basic fact of Model Theory that the $\mathcal{L}_{v_f}(F)$ -type $\text{tp}(\mathbf{a})$ is closed under $\mathcal{L}_{v_f}(F)$ -automorphisms. Thus $\text{Orb}(\mathbf{a}) \subseteq \text{tp}(\mathbf{a})$. By the second part of this theorem, $\text{Orb}(\mathbf{a})$ is $\mathcal{L}_{\text{ring}}(F)$ -definable; thus $\text{tp}(\mathbf{a}) \subseteq \text{Orb}(\mathbf{a})$.

□

6. SUBSETS AND SUBFIELDS OF $F((t))$

Suppose that X is an F -definable **subset** of $F((t))$, i.e. $X \subseteq F((t))$, and let (X) denote the subfield of $F((t))$ generated by X .

Proposition 18. *If $X \not\subseteq F$ then $X \setminus F$ is a union of infinite $\exists\text{-}\mathcal{L}_{\text{ring}}(F(t))$ -definable sets.*

Proof. Let $a \in X \setminus F$. Then a is of transcendence degree 1 over F . By Theorem 1, $\text{Orb}(a) \subseteq X$ is infinite and $\exists\text{-}\mathcal{L}_{\text{ring}}(F(t))$ -definable. □

Corollary 2. *If $X \not\subseteq F$ then there exists $n \in \mathbb{N}$ such that $(X) = F((t^{p^n}))$.*

Proof. In [2] it was shown that this result holds for existentially definable sets (even with parameters). By Proposition 18, X contains an infinite existentially definable set. □

In particular, if the field of constants is finite, we have the following corollary.

Corollary 19. *If $F = \mathbb{F}_p$ then either $(X) = \mathbb{F}_p$ or there exists $n \in \mathbb{N}$ such that $(X) = \mathbb{F}_p((t^{p^n}))$.*

6.1. Subsets and subfields of $F((t))^{\text{perf}}$. Suppose now that X is an F -definable subset of $F((t))^{\text{perf}}$, i.e. $X \subseteq F((t))^{\text{perf}}$.

Corollary 20. *If $X \not\subseteq F$ then $(X) = F((t))^{\text{perf}}$.*

Proof. Let $x \in X \setminus F$. Let $n \in \mathbb{Z}$ be chosen maximal such that $x \in F((t^{p^n}))$. Then $x \notin F((t^{p^{n+1}}))$. Let $s := t^{p^n}$. The set $X \cap F((s))$ is invariant under F -automorphisms of $F((s))$, since automorphisms of $F((s))$ extend to automorphisms of $F((s))^{\text{perf}}$. By Proposition 18, $X \cap F((s))$ contains an infinite $\exists\text{-}F(s)$ -definable set. As before we apply the result from [2], thus the field generated by $X \cap F((s))$ is $F((s))$. In particular, $F((s)) \subseteq (X)$.

Now consider the automorphism f of $F((t))^{\text{perf}}$ that fixes F pointwise and sends $t \mapsto t^{1/p}$. The set X is closed under f . Thus $(X) = F((t))^{\text{perf}}$, as required. □

Remark 21. These results can be seen in the context of Corollary 5.6, from [5], in which it is shown that a henselian field of characteristic zero has no proper parameter-definable subfields; and Question 10 of [5] in which Junker and Koenigsmann ask whether $\mathbb{F}_p((t))^{\text{perf}}$ is very slim (see Definition 1.1 in [5]). If $\mathbb{F}_p((t))^{\text{perf}}$ were very slim then in particular it would have no infinite proper parameter-definable subfields. Corollary 20 shows that $\mathbb{F}_p((t))^{\text{perf}}$ has no infinite proper subfields which are \emptyset -definable but at present we are not able to extend our methods to study sets definable with parameters.

7. APPENDIX: DEFINABILITY OF CERTAIN SUBSETS OF $F((t))$

The following well-known fact is based on an old result of Julia Robinson about the p -adic numbers. The original statement can be found in Section 2 of [8].

Fact 22 (Folklore, based on [8]). *The valuation ring $F[[t]]$ is defined by the $\mathcal{L}_{\text{ring}}(t)$ -formulas:*

- (1) $A(x; t) := \exists y \ 1 + x^l t = y^l$ (for some prime $l \neq p$), and
- (2) $B(x; t) := \neg \exists z \ (xzt = 1 \wedge A(z; t))$.

The next lemma collects together several well-known and easy consequences of Fact 22. For the convenience of the reader, in the following two lemmas we write ‘ (\exists) ’ or ‘ (\forall) ’ after each formula to denote whether the formula is existential or universal in the given language.

Lemma 23 (Definitions in $\mathcal{L}_{\text{ring}}(t)$). *We have that \mathcal{M} is defined by the $\mathcal{L}_{\text{ring}}(t)$ -formulas:*

- (5) $C(x; t) := \exists y \ (x = 0 \vee (xy = 1 \wedge \neg B(y; t)))$ (\exists) , and
- (6) $D(x; t) := \neg \exists y \ (xy = 1 \wedge A(y; t))$ (\forall) ;

and \mathcal{O}^\times is defined by the $\mathcal{L}_{\text{ring}}(t)$ -formulas:

- (7) $E(x; t) := \exists y \ (A(x; t) \wedge A(y; t) \wedge xy = 1)$ (\exists) , and
- (8) $F(x; t) := \neg \exists y \ (C(x; t) \vee (yx = 1 \wedge C(y; t)))$ (\forall) ;

and \mathcal{U} is defined by the $\mathcal{L}_{\text{ring}}(t)$ -formulas:

- (10) $G(x; t) := \exists y \ (E(y; t) \wedge x = yt)$ (\exists) , and
- (11) $H(x; t) := \forall y \forall z \ (D(x; t) \wedge \neg(x = yz \wedge C(y; t) \wedge C(z; t)))$ (\forall) .

For convenience, this next lemma collects some well-known facts about some \mathcal{L}_{vf} -definable subsets of $F((t))$.

Lemma 24 (Definitions in \mathcal{L}_{vf}). *We have that \mathcal{M} is defined by the \mathcal{L}_{vf} -formulas:*

- (1) $C'(x) := \exists y \ (x = 0 \vee (xy = 1 \wedge y \notin \mathcal{O}))$ (\exists) , and
- (2) $D'(x) := \neg \exists y \ (xy = 1 \wedge y \in \mathcal{O})$ (\forall) ;

and \mathcal{O}^\times is defined by the \mathcal{L}_{vf} -formulas:

- (3) $E'(x) := \exists y \ (x \in \mathcal{O} \wedge y \in \mathcal{O} \wedge xy = 1)$ (\exists) , and
- (4) $F'(x) := \neg \exists y \ (C'(x) \vee (yx = 1 \wedge C'(y)))$ (\forall) ;

and \mathcal{U} is defined by the \mathcal{L}_{vf} -formula:

- (9) $H'(x) := \forall y \forall z \ (D'(x) \wedge \neg(x = yz \wedge C(y) \wedge C(z)))$ (\forall) ;

The following fact is due to James Ax and is found in the proof of the Theorem in [3].

Fact 25 ([3]). *The valuation ring $F[[t]]$ is defined by the $\mathcal{L}_{\text{ring}}$ -formula:*

$$A''(x) := \exists w \exists y \forall u \forall x_1 \forall x_2 \exists z \forall y_1 \forall y_2 ((z^m = 1 + wx_1^m x_2^m \vee y_1^m \neq 1 + wx_1^m \vee y_2^m \neq 1 + wx_2^m) \wedge u^m \neq w \wedge y^m = 1 + wx^m).$$

Lemma 26 (Definitions in $\mathcal{L}_{\text{ring}}$). \mathcal{M} , \mathcal{O}^\times , and \mathcal{U} are $\mathcal{L}_{\text{ring}}$ -definable.

Proof. Let $A''(x)$ be as in Fact 25. For any variable u we replace the atomic formula $u \in \mathcal{O}$ with $A''(u)$ in the \mathcal{L}_{vf} -formulas $C'(x)$, $E'(x)$, and $H'(x)$ to obtain $\mathcal{L}_{\text{ring}}$ -formulas $C''(x)$, $E''(x)$, and $H''(x)$. \square

ACKNOWLEDGEMENTS

The author would like to thank his doctoral supervisor Jochen Koenigsmann for all his helpful guidance, and Arno Fehm, Immanuel Halupczok, Franziska Jahnke, and Dugald Macpherson for helpful discussions.

REFERENCES

- [1] Will Anscombe. *Definability in Henselian Fields*. PhD thesis, University of Oxford, 2012.
- [2] Will Anscombe. \exists -definability in t -henselian fields. *Manuscript*, 2015.
- [3] James Ax. On the undecidability of power series fields. *Proc. Amer. Math. Soc.*, 16:846, 1965.
- [4] James Ax and Simon Kochen. Diophantine problems over local fields III. *Ann. Math.*, 83:437–456, 1966.
- [5] Markus Junker and Jochen Koenigsmann. Schlanke Körper. *J. Symb. Logic*, 75:481–500, 2010.
- [6] Saunders Mac Lane. Subfields and automorphism groups of p -adic fields. *Ann. Math.*, 40:423–442, 1939.
- [7] Alexander Prestel and Martin Ziegler. Model-theoretic methods in the theory of topological fields. *J. Reine Angew. Math.*, 299 (300):318–341, 1978.
- [8] Julia Robinson. The decision problem for fields. In *Theory of Models (Proc. 1963 Internat. Sympos. Berkeley)*, Lecture Notes in Computer Science, pages 299–311, Amsterdam, 1965. North-Holland.
- [9] O. F. G. Schilling. Automorphisms of fields of formal power series. *Bull. Amer. Math. Soc.*, 50:892–901, 1944.

SCHOOL OF MATHEMATICS, UNIVERSITY OF LEEDS, LEEDS LS2 9JT, UNITED KINGDOM
E-mail address: pmtwga@leeds.ac.uk